§11.8 Power Series:

Motivation: The standard Geometric Sum for $1+x+x^2+x^3+\cdots$

$$\sum_{n=0}^{\infty} x^n \begin{cases} \frac{1}{1-x} & \text{conv for } |x|<1. \\ \text{DIV} & \text{for } |x| \geq 1. \end{cases}$$

Previously, $\sum a_n$'s nth term a_n are considered $\boxed{\text{FIXED NUMBERS}}$. In the rest of the chapter, we want to consider $\sum a_n$ and a_n as $\boxed{\text{function of } x.}$ Like above, $a_n = x^n$.

Power Series:

$$\sum_{n=0}^{\infty} c_n \cdot (x-a)^n = c_0 + c_1\cdot(x-a)^1 + c_2\cdot(x-a)^2 + c_3\cdot(x-a)^3 + \cdots + c_n\cdot(x-a)^n + \cdots$$

where a is a constant, $c_0, c_1, c_2, \cdots$ is a sequence. This series is a function of x, called Power Series. (each term is a power function of x). We also call a the center and c_n the coefficients.

e.g. 1. Find the center and coefficients of the following power series.

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1\cdot(x-0)^n, \quad a=0, \quad c_0=c_1=c_2=\cdots=1 \quad (c_n=1 \text{ for all } n)$$

(s16) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^k}{3^k\cdot(k+1)}$, $a=0$, $c_k = \dfrac{(-1)^k}{3^k\cdot(k+1)}$, $k=0,1,2,\cdots$

★ (s15) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$ $\quad \dfrac{(2x+1)^n}{n} = \dfrac{[2(x+\frac{1}{2})]^n}{n} = \dfrac{2^n}{n}\cdot[x-(-\tfrac{1}{2})]^n$, $\boxed{a=-\frac{1}{2}}$, $c_n = \boxed{\dfrac{2^n}{n}}$

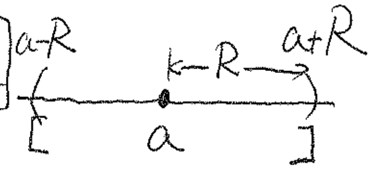
Remark: center a can be found by letting $2x+1=0 \Rightarrow x=-\frac{1}{2}$.

★ (f14) $-3+9(x+5)-27\cdot(x+5)^2+81\cdot(x+5)^3 - \cdots$

$$= \sum_{n=0}^{\infty}(-3)^{n+1}\cdot(x+5)^n, \quad a=-5, \quad c_n=(-3)^{n+1}, \; n=0,1,2,\cdots$$

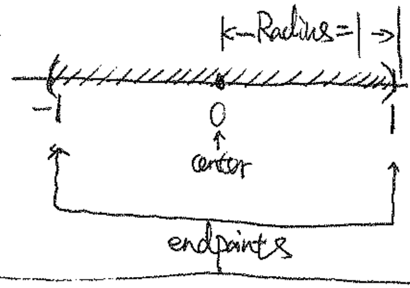
Goal: We want to know FOR WHICH VALUES (of x) does $\sum c_n\cdot(x-a)^n$ converge?

Method: Let $a_n = c_n\cdot(x-a)^n$ and apply Ratio Test to determine the $\boxed{\text{RADIUS of Convergence } R}$ and $\boxed{\text{INTERVAL of Convergence}}$



e.g.2 (Trivial example) $\sum_{n=0}^{\infty} x^n$. According to standard Geometric Series, $\sum_{n=0}^{\infty} x^n$ is $\boxed{\text{convergent if } |x|<1}$ and divergent if $|x| \geq 1$.

$|x|<1$ represents $-1<x<1$, i.e., the following interval



Conclusion: Radius of Conv: $R=1$.

Interval of Conv: $(-1, 1)$ (both sides are open).

(Non-Trivial examples by ratio test)

e.g.3 Find the radius of convergence of $\sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^k}{3^k (k+1)}$

(s/6, M.C)

Solution: Let $a_k = \frac{(-1)^k \cdot x^k}{3^k \cdot (k+1)}$ and apply (full version) Ratio Test.

$a_{k+1} = \frac{(-1)^{k+1} \cdot x^{k+1}}{3^{k+1} \cdot (k+2)}$, $\left|\frac{a_{k+1}}{a_k}\right| = \left| \frac{\frac{(-1)^{k+1} \cdot x^{k+1}}{3^{k+1} \cdot (k+2)}}{\frac{(-1)^k \cdot x^k}{3^k \cdot (k+1)}} \right| = \left| \frac{(-1)^{k+1} \cdot x^{k+1}}{3^{k+1} \cdot (k+2)} \cdot \frac{3^k \cdot (k+1)}{(-1)^k \cdot x^k} \right|$

$= \left| \frac{(-1)^{k+1}}{(-1)^k} \cdot \frac{x^{k+1}}{x^k} \cdot \frac{3^k (k+1)}{3^{k+1}(k+2)} \right| = \left| (-1) \cdot x \cdot \frac{k+1}{3(k+2)} \right| = \frac{k+1}{3(k+2)} \cdot |x|$.

$\boxed{\text{Caution:}}$ ☆☆☆ x can be both positive and negative. DO NOT drop the abstract value for x.

$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k+1}{3(k+2)} \cdot |x| = \frac{1}{3}|x|$ (x is fixed, limit does not effect x).

☆ $\boxed{\text{THEN SET ABOVE LIMIT} < 1}$, i.e., $\boxed{\frac{1}{3}|x| < 1} \Rightarrow \boxed{|x| < 3}$ ← Radius of Conv.

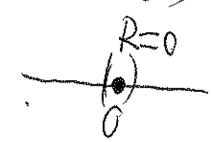
$\boxed{\text{Radius of Conv } R = 3}$.

Remark: Radius of Convergence might be $R=0$ or $R=+\infty$.

e.g.4 Find radius of Conv for the following power series

- $\sum n!(2x)^n$, $a_n = n!(2x)^n$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(n+1)!(2x)^{n+1}}{n!(2x)^n} \right| = \lim_{n\to\infty} |(n+1) \cdot (2x)| = \infty$ (unless $x=0$)

  Except $x=0$, for all other values of x, $\lim = \infty > 1$. $\boxed{R=0}$



- $\sum \frac{(x-1)^n}{n!}$, $a_n = \frac{(x-1)^n}{n!}$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x-1)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)}{n+1} \right| = 0$ (for all x)

The limit $= 0 < 1$ $\boxed{\text{for all } x}$, $\boxed{R=\infty}$

☆ e.g 5. (Radius + INTERVAL).

(S15). Find $\boxed{\text{all values of } x}$ for which the series $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$ converges.
14pts

Solution: • (Step 1: Ratio Test for Radius of Conv)

$$a_n = \frac{(2x+1)^n}{n}, \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x+1)^{n+1}}{(n+1)} \cdot \frac{n}{(2x+1)^n}\right| = \lim_{n\to\infty}\left|(2x+1)\cdot\frac{n}{n+1}\right| = |2x+1|$$ (KEEP the abstract value)

Set the limit to be less than 1. $|2x+1| < 1$ ($R = \frac{1}{2}$ since $|x+\frac{1}{2}| < \frac{1}{2}$).

Solve the inequality for x. $\boxed{-1 < 2x+1 < 1}$ Caution: Two sided inequality when Abs is removed.

i.e. $-2 < 2x < 0 \Rightarrow \boxed{-1 < x < 0}$, $x \in (-1, 0)$

Two endpoints are $x=-1$, $x=0$.

• (Step 2: Test endpoints)

at $x = 0$, $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n} \overset{x=0}{=\!=} \sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, p-Series, $p=1$, DIV.

$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$ is DIV at $x=0$, therefore, $x=0$ is NOT inclued (open).

at $x=-1$, $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n} \overset{x=-1}{=\!=} \sum_{n=1}^{\infty} \frac{(-2+1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, alternating series with $b_n = \frac{1}{n}$.

$\lim_{n\to\infty} b_n = \lim \frac{1}{n} = 0$ and $\frac{1}{n}$ is decreasing, therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent.

$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$ is Conv at $x=-1$, therefore, $x=-1$ is inclued (closed)

So the series $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n}$ converges on $[-1, 0)$, (or converges for $x \in [-1, 0)$)

(or converges for $-1 \le x < 0$)

Conclusion: Find all values of x for which $\sum C_n (x-a)^n$ converges.

Step 1: Set $a_n = C_n \cdot (x-a)^n$. Compute $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$

Step 2: Set limit $L < 1$, we have such inequality $C|x-a| < 1$, $\Leftrightarrow |x-a| < \boxed{\frac{1}{C} = R}$

Solve for x, we have an interval $(a - \frac{1}{C}, a + \frac{1}{C})$

Step 3: Test the two endpoints $x = a - \frac{1}{C}$ and $x = a + \frac{1}{C}$ (separately).

Usually, one endpoint will need A.S. Test.

e.g. 6 Consider $-3+9(x+5)-27\cdot(x+5)^2+81\cdot(x+5)^3$

(f14) (a). Write the series in $\sum_{n=0}^{\infty} a_n$ (b). What is the center? (c). Find the interval of Conv!

(a). (eg. 1) $a_n = (-3)^{n+1}\cdot(x+5)^n$ (double check your expression by plugging $n=0,1,2,3$)
$n=0,1,2,\dots$

i.e. $-3+9(x+5)-27\cdot(x+5)^2+81\cdot(x+5)^3+\dots = \sum_{n=0}^{\infty}(-3)^{n+1}\cdot(x+5)^n$.

(b). Center $a=-5$. (i.e. $(x+5)^n = [x-\underbrace{(-5)}_{a}]^n$, or solve for $x+5=0 \Rightarrow x=-5$)

(c). Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+2}\cdot(x+5)^{n+1}}{(-3)^{n+1}\cdot(x+5)^n}\right| = \left|(-3)\cdot(x+5)\right| = 3|x+5|$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3\cdot|x+5| < 1 \Rightarrow |x+5|<\frac{1}{3} \Rightarrow -\frac{1}{3}<x+5<\frac{1}{3}$

$\Rightarrow -5-\frac{1}{3} < x < -5+\frac{1}{3} \Rightarrow$ $\boxed{-\frac{16}{3} < x < -\frac{14}{3}.\ \text{endpoints } x=-\frac{16}{3},\ x=-\frac{14}{3}}$

Test endpoints:

$x=-\frac{16}{3}$, $\sum_{n=0}^{\infty}(-3)^{n+1}\cdot\left(-\frac{16}{3}+5\right)^n = \sum_{n=0}^{\infty}(-3)^{n+1}\cdot\left(-\frac{1}{3}\right)^n$ Hint: $(-3)^{n+1}\left(-\frac{1}{3}\right)^n = (-3)^{n+1}\cdot\frac{1}{(-3)^n}$

$= \sum_{n=0}^{\infty}(-3)$. DIV. (constant series) $= (-3)^{n+1-n}$

$x=-\frac{14}{3}$. $\sum_{n=0}^{\infty}(-3)^{n+1}\cdot\left(-\frac{14}{3}+5\right)^n = \sum_{n=0}^{\infty}(-3)^{n+1}\cdot\left(\frac{1}{3}\right)^n$ $= -3$.

Hint: $(-3)^{n+1} = (-1)^{n+1}\cdot 3^{n+1}$

$= \sum_{n=0}^{\infty} +3\cdot(-1)^{n+1}$ DIV. (DIV test)

i.e., both endpoints are divergent.

Therefore, the interval of convergence of $\sum_{n=0}^{\infty}(-3)^{n+1}\cdot(x+5)^n$ is $\boxed{\left(-\frac{16}{3}, -\frac{14}{3}\right)}$.

Remark 1: Abs inequality $|\square\cdot x+\triangle| < \bigcirc$ yields two sided ineq. $-\bigcirc < \square\cdot x+\triangle < \bigcirc$

Remark 2: For $\sum_{n=0}^{\infty} c_n\cdot(x-a)^n$, a ~~directly~~ direct formula for Radius of Conv is

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|}.$$ (Can be derived via Ratio Test)

Remark 3: If $R=0$, then interval of Conv is $\{a\}$ (only the center point) $(x=a)$

If $R=\infty$, then interval of Conv is $(-\infty,+\infty)$ (for all x).